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THE INJECTIVITY, PROJECTIVITY OF MODULES AND SOME RELATED CLASSES OF RINGS

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INTRODUCTION

1. Motivation for the study

The concept of injective modules was first introduced by Baer [4] in 1940. A module M is called injective if every homomorphism from a submodule A of a module N to M can be extended to a homomorphism from N to M. The dual notion to injectivity is projectivity, which was introduced by Cartan and Eilenberg [8] in 1956. A module M is projective if, for all modules A and submodules $X \leq A$, every homomorphism $\varphi : M \to A/X$ can be lifted to a homomorphism $\phi : M \to A$.

One of the extensions of the class of injective modules is the class of quasi-injective modules. This class was introduced by Johnson and Wong [24] in 1961 and consists of modules that are invariant under all endomorphisms of their injective hulls. The dual notion of quasi-injective modules is quasi-projective modules, which was introduced by Wu and Jans [51] in 1967. A module M is said to be quasi-projective if, for every submodule $X \leq M$, every homomorphism $f : M \to M/X$ can be lifted to an endomorphism of M.

In 1967, Singh and Jain [43] introduced the concept of pseudo-injective modules, which extends the class of quasi-injective modules. A module M is said to be pseudo-injective if for every submodule N of M, each monomorphism from N to M can be extended to an endomorphism of M. In 1969, Dickson and Fuller [12] investigated the class of indecomposable quasi-injective modules. Building on this result, Lee and Zhou [32] introduced the notion of automorphism-invariant modules in 2013. A module is said to be automorphism-invariant if it is invariant under all automorphisms of its injective hull. Then, Er, Singh, and Srivastava [13] showed that the class of automorphism-invariant modules coincides with the class of pseudo-injective modules. The dual notion of pseudo-injective modules is pseudo-projective modules. In [46], a module M is called pseudo-projective if, for every module A and every pair of epimorphisms $f, g : M \to A$, there exists an endomorphism h of M such that $f = g \circ h$. Tiwary and Pandeya [46] used the pseudo-projective property to characterize semisimple Artinian rings and right perfect rings. In 2024, Trang and Dung [47] studied pseudo-projective modules to characterize semisimple Artinian rings and formal triangular matrix rings. However, research on the characterization of other classical ring classes such as hereditary rings, semilocal rings or Artinian principal ideal rings via pseudo-projective modules remains limited. Motivated by this, the thesis focuses on studying the pseudo-projective property of modules to characterize certain classical classes of rings.

Extending the research on automorphism-invariant modules, Koşan and Quynh [26] presented the concept of nilpotent-invariant modules in 2017. A module M is said to be nilpotent-invariant if it is invariant under all nilpotent endomorphisms of its injective hull. In 2021, Quynh, Abyzov and Tai [37] further developed the properties and dualities associated with this concept. In this dissertation, we continue the investigation of nilpotent-invariant modules, focusing particularly on their behavior over formal triangular matrix rings.

In 1969, Jain, Mohamed and Singh [22] investigated the ring structure through the quasi-injectivity of one-sided ideals. They defined a class of rings in which every right ideal is quasi-injective, termed right q-rings and used this to characterize semisimple Artinian rings. In 2014, Singh and Srivastava [44] introduced right a-rings, rings in which every right ideal is automorphism-invariant. Later, in 2016, Koşan, Quynh and Srivastava [27] explored the structure of right a-rings. Naturally, this motivates the consideration of rings in which every right ideal is nilpotent-invariant. We refer to such rings as right $\mathfrak{n}$-rings.

Building upon the study of right q-rings, Hill [20] proved that every semiperfect right q-ring is a direct sum of a basic ring and a semisimple Artinian ring. In 2022, Quynh, Abyzov and Trang [38] investigated a class of rings in which every finitely generated right ideal is automorphism-invariant and referred to such rings as right fa-rings. The authors described the structure of right fa-rings in terms of formal triangular matrix rings.

From the property of automorphism-invariant rings that every primitive idempotent is local, it follows that any ring which does not contain an infinite set of orthogonal idempotents and is automorphism-invariant must be semiperfect. However, the behavior of automorphism-invariant ideals in the class of semiperfect right a-rings, especially under I-finite condition, has not yet been fully investigated. This observation motivates the study of right I-finite a-rings through the structure of basic semiperfect rings. This is precisely the focus of our investigation.

In this dissertation, we investigate the ring structures through the pseudo-projectivity of modules and examine formal triangular matrix rings with the condition of nilpotent-invariance of modules. In addition, several characterizations of ring classes based on the automorphism-invariance and nilpotent-invariance of one-sided ideals have also been studied and developed.

Based on the results obtained concerning the classes of pseudo-projective modules, automorphism-invariant modules and nilpotent-invariant modules, we observe that, in order to study pseudo-projective modules in characterizing rings, investigate nilpotent-invariant modules over formal triangular matrix rings and examine rings in which every ideal is automorphism-invariant or nilpotent-invariant, it is necessary to understand the class of injective modules and their extensions, as well as their dual notions. Accordingly, the topic of this dissertation is formulated as: "**The injectivity, projectivity of modules and some related classes of rings**".

2. Aims of the study

This dissertation is conducted with the following aims:

- To characterize classical rings (such as perfect, semiperfect, hereditary, semihereditary, semilocal and Artinian principal ideal rings) via peudo-projective modules.

- To analyze the structure of right I-finite a-rings in which the right and left socles coincide and are essential in R_R.

- To investigate rings in which every right ideal is nilpotent-invariant and to study nilpotent-invariant modules over formal triangular matrix rings.

3. Object and scope of the study

The objects of this thesis include:

- Several important module classes, including pseudo-projective, automorphism-invariant, and nilpotent-invariant types.

- Classical classes of rings, including right perfect, semiperfect, right hereditary, right semihereditary, semilocal and Artinian principal ideal rings. Additionally, this study considers ring classes in which every right ideal is automorphism-invariant or nilpotent-invariant, as well as formal triangular matrix rings.

The scope of the study is restricted to the category of right (left) modules over associative rings with identity. The rings considered belong to classical classes as mentioned above, as well as right a-rings, $\mathfrak{n}$-rings and formal triangular matrix rings. In addition, the thesis also investigates the structure and properties of pseudo-projective, automorphism-invariant and nilpotent-invariant modules.

4. Research methodology

This research belongs to the field of fundamental mathematics. From the start, we reviewed specialized literature and research papers related to pseudo-projective modules, automorphism-invariant modules, nilpotent-invariant modules, as well as classical rings and right a-rings. Then, we analyzed and synthesized the materials, engaged in scholarly discussions with the thesis advisor and experts in the field and developed the research problems through seminars and workshops. We used methods such as generalization, comparison, logical reasoning and classification in order to formulate the research problems addressed in this thesis.

5. Scientific and practical significance of the thesis

The thesis provides characterizations of several classical classes of rings via the pseudo-projectivity of modules; further investigates and extends the class of right a-rings with I-finite condition and introduces a new class of rings called right $\mathfrak{n}$-rings. The results concerning properties, characterizations and structural descriptions, as well as the relationships between right $\mathfrak{n}$-rings and other classes of rings, contribute meaningfully to the enrichment of ring theory. With the results obtained, this research holds significant importance in the study of ring and module theory. It enriches the characterizations of classical classes of rings and contributes to motivate and promote further interest in ring and module theory among researchers.

6. Structure of the thesis

The thesis consists of three chapters. Chapter 1 presents fundamental concepts and results in ring and module theory that are necessary for the developments in Chapters 2 and 3. In Chapter 2, we characterize various classes of rings such as semiperfect rings, right perfect rings, right semihereditary rings, right hereditary rings, semilocal rings and Artinian principal ideal rings via pseudo-projective modules. The main result of this chapter is Theorem 2.3.3, which asserts that a ring is Artinian principal ideal if and only if the class of pseudo-injective modules coincides with the class of pseudo-projective modules. Chapter 3 investigates the structure of right I-finite a-rings and establishes conditions under which such rings are quasi-Frobenius (Theorem 3.1.13). We introduce the concept of a right $\mathfrak{n}$-ring, defined as a ring in which every right ideal is nilpotent-invariant. Various properties and decompositions of such rings are studied. In particular, every right $\mathfrak{n}$-ring is shown to be the direct sum of a square-full semisimple Artinian ring and a right square-free ring (Theorem 3.2.13). Furthermore, nilpotent-invariant modules over formal triangular matrix rings are also considered in this chapter.

CHAPTER 1. PRELIMINARIES

In this chapter, we recall the fundamental definitions and some basic properties that will be used throughout the thesis.

1.1. Notations and concepts

Given a ring R, we denote by M_R (respectively, ${}_RM$) that M is a right (respectively, left) R-module. Throughout this thesis, when referring to an R-module M, we assume it to be a right R-module and simply write M instead of M_R. For R-modules M and N, we denote by $\mathrm{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. In particular, The endomorphism set of the R-module M is denoted by $\mathrm{End}_R(M)$, or simply $\mathrm{End}(M)$. The Jacobson radical and the socle of a module M are denoted by $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$, respectively. In particular, $J(R)$ denotes the Jacobson radical of the ring R, while $\mathrm{Soc}(R_R)$ and $\mathrm{Soc}({}_RR)$ denote the right and left socles of the ring R, respectively.

Let M be a right R-module and $\emptyset \neq X \subseteq M$. The *right annihilator* of X in R, denoted by $r_R(X)$, is defined by

$$r_R(X) = \{r \in R \mid xr = 0, \text{ for all } x \in X\}.$$

When $X = \{x\}$, we write $r_R(x)$ instead of $r_R(\{x\})$. Clearly, $r_R(X)$ is a right ideal of the ring R. Moreover, if X is a submodule of M then $r_R(X)$ is an ideal of R. The *left annihilator* of X in R, denoted by $l_R(X)$, is defined similarly [1, pp. 172-173].

A submodule K of M is called *essential* (or *large*) in M, denoted by $K \leq^e M$, if for every submodule $L \leq M$, the condition $K \cap L = 0$ implies $L = 0$. A monomorphism $\alpha : M \to Q$ is called an *essential monomorphism* if $\mathrm{Im}(\alpha) \leq^e Q$. Dually, a submodule

K of M is called *superfluous* (or *small*) in M, denoted by $K \ll M$, if for every submodule $L \leq M$, the condition $K + L = M$ implies $L = M$. An epimorphism $\alpha : M \to Q$ is called a *superfluous epimorphism* if $\mathrm{Ker}(\alpha) \ll M$ [25, Definition 5.1.1].

A module M is called *square-free* if it does not contain a direct sum of two isomorphic nonzero submodules. A module M is called *square-full* if every non-zero submodule of M contains a non-zero square root in M [33, Definition 2.34].

A module P is called *finitely presented* if there exists a exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$, where F is a finitely generated free module and K is a finitely generated module [30, Definition 4.25].

A module M is called *semiprimitive* if $\mathrm{Rad}(M) = 0$ [21, pp. 107].

A module M is said to have the *internal cancellation property* if whenever $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$ [33, Definition 1.22].

A ring R is called *stably-finite* if the matrix ring $\mathbb{M}_n(R)$ is directly-finite for all $n \geq 1$ [30, pp. 5].

1.2. Extensions of injective and projective modules

A module M is called *N-injective* if, for every submodule $A \leq N$, any homomorphism $f : A \to M$ can be extended to a homomorphism $\overline{f} : N \to M$; that is, the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & N \arrow[dl, dashed, "\bar{f}"] \\
& M &
\end{tikzcd}$$

A module M is called *injective* if it is N-injective for every module N. In addition, Baer provided a criterion for testing the injectivity of modules, commonly referred to as the "Baer Criterion". According to this criterion, a module M is injective if and only if it is R_R-injective [34, pp. 6, 8].

For a module M, a monomorphism $\alpha : M \to Q$ is called an *injective hull* of M if Q is an injective module and α is an essential monomorphism. The injective hull of a module M is denoted by $E(M)$ [34, pp. 6].

A module M is called *automorphism-invariant* if $f(M) \leq M$ for every automorphism f of its injective hull $E(M)$ [32, Definition 1].

A module M is called *nilpotent-invariant* if $f(M) \leq M$ for every nilpotent endomorphism f of its injective hull $E(M)$. A ring R is called a *right nilpotent-invariant ring* if R_R is a nilpotent-invariant module [26, pp. 2776].

A module M is called *A-projective* if, for every submodule $X \leq A$, any homomorphism $\varphi : M \to A/X$ can be lifted to a homomorphism $\phi : M \to A$; that is, the following diagram commutes:

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "\phi"'] \arrow[d, "\varphi"] & \\
A \arrow[r] & A/X \arrow[r] & 0
\end{tikzcd}$$

A module M is called *projective* if it is A-projective for every module A. A module M is called *quasi-projective* if it is M-projective [33, Definition 4.29].

A module M is called *pseudo-projective* if for every module A and every pair of epimorphisms $f, g : M \to A$, there exists a homomorphism $h : M \to M$ such that $f = g \circ h$; that is, the following diagram is commutative [46, Definition 1.1]

$$\begin{tikzcd} & M \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\ M \arrow[r, "g"] & A \arrow[r] \arrow[d] & 0. \\ & 0 & \end{tikzcd}$$

A module P is called a *projective cover* of a module M if there exists an epimorphism $p : P \to M$ such that P is a projective module and $\mathrm{Ker}(p) \ll P$ [34, pp. 261]. A module P is called a *pseudo-projective cover* of a module M if there exists an epimorphism $p : P \to M$ such that P is a pseudo-projective module and $\mathrm{Ker}(p) \ll P$ [47, pp. 1399].

1.3. Some related classes of ring

Definition 1.3.5 ([1, pp. 146])**.** A ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian.

Definition 1.3.6 ([1, pp. 150])**.** A ring R is called *semiperfect* if R is semilocal and every idempotent lifts modulo $J(R)$.

Definition 1.3.7 ([20, pp. 114])**.** Let R be a semiperfect ring and let $e \in R$ be a primitive idempotent. We say that e has *index* h if in every decomposition of R_R into a direct sum of indecomposable modules, there are exactly h direct summands isomorphic to eR.

If R is a semiperfect ring then there exists a complete set of pairwise orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$. Suppose that $\{e_iR/e_iJ(R) \mid 1 \leq i \leq n\}$ is a complete set of representatives of isomorphism classes of simple right R-modules. Then the set $\{e_1, e_2, \ldots, e_n\}$ is called a *basic set of idempotents* of R. If $e = e_1 + e_2 + \cdots + e_n$ then the ring eRe is called the *basic ring* of R. A ring R is called *basic semiperfect* if $m = n$; that is, $1 = e_1 + e_2 + \cdots + e_n$, where $\{e_i\}_{i=1}^{n}$ is a basic set of local idempotents [34, pp. 62, 260].

Definition 1.3.12 ([34, pp. 269])**.** A ring R is called *right perfect* if R is semiperfect and the ideal $J(R)$ is right T-nilpotent.

Definition 1.3.15 ([16, pp. 340])**.** A ring R is called *right hereditary* if every right ideal of R is projective, equivalently, if every submodule of a projective right R-module is projective. A ring R is called *hereditary* if it is both left and right hereditary. A ring R is called *right semihereditary* if every finitely generated right ideal of R is projective, equivalently, if every finitely generated submodule of a projective right R-module is projective. A ring R is called *semihereditary* if it is both left and right semihereditary.

Definition 1.3.17 ([36, pp. 547])**.** A ring R is called a *right S-ring* if every finitely generated flat right R-module is projective.

Definition 1.3.19 ([7, pp. 72-73])**.** A right R-module M is called *coherent* if every finitely generated submodule of M is finitely presented. A ring R is called *right coherent* if R_R is coherent. A ring R is called *right* Π-*coherent* if every finitely generated submodule of ΠR_R is finitely presented. A ring R is called Π-*coherent* if it is both left and right Π-coherent.

Definition 1.3.22 ([1, pp. 129])**.** A ring R is called *semiprimary* if R is semilocal and $J(R)$ is a nilpotent ideal.

Definition 1.3.25 ([34, pp. 20])**.** A ring is called *quasi-Frobenius* if it is left and right Artinian and left and right self-injective.

According to Faith and Walker, a ring is quasi-Frobenius if and only if every projective module is injective and vice versa.

Theorem 1.3.26 ([14, Theorem 24.20]). *The following conditions are equivalent for a ring R:*

(1) R is a quasi-Frobenius ring.

(2) Every injective right R-module is projective.

(3) Every projective right R-module is injective.

From the results of Fuller, Faith and Byrd ([6], [14], [15]), we conclude that, over a Artinian principal ideal ring, the class of quasi-projective modules coincides with the class of dual modules. We recall this result in the following theorem.

Theorem 1.3.27 ([6], [14], [15]). *The following conditions are equivalent for a ring R:*

(1) R is an Artinian principal ideal ring.

(2) Every quasi-injective right R-module is quasi-projective.

(3) Every quasi-projective right R-module is quasi-injective.

(4) Every quotient ring of R is a quasi-Frobenius ring.

Definition 1.3.40 ([44, pp. 13]). A ring R is said to be a *right a-ring* if each right ideal of R is automorphism-invariant.

CHAPTER 2. THE CLASS OF PSEUDO-PROJECTIVE MODULES AND RELATED RINGS

The aim of this chapter is to describe pseudo-projective modules over some classical classes of rings.

2.1. Pseudo-projective modules over semiperfect and perfect rings

Perfect rings can be characterized through pseudo-projective covers of modules as follows:

Theorem 2.1.5. *A ring R is right perfect if and only if, for every semisimple right R-module M, the module $M \oplus F$ has a pseudo-projective cover for each free right R-module F.*

The following theorem provides a characterization of right S-rings via the pseudo-projectivity and quasi-projectivity properties of modules.

Theorem 2.1.6. *The following statements are equivalent for a ring R:*

(1) R is a right S-ring.

(2) Every finitely generated flat right R-module is quasi-projective.

(3) Every finitely generated flat right R-module is pseudo-projective.

Definition 2.1.8. A right R-module M is called a Π-*pseudo-projective* module if ΠM_R is a pseudo-projective right R-module.

A characterization of rings is also obtained via the pseudo-projectivity of direct products of projective (or flat) modules.

Theorem 2.1.9. *The following conditions are equivalent for a ring R:*

(1) *R is right perfect left coherent.*

(2) *The direct product of any family of copies of R is projective as a right R-module.*

(3) *All direct products of projective right R-modules are pseudo-projective.*

(4) *All direct products of flat right R-modules are pseudo-projective.*

(5) *Every projective right R-module is Π-pseudo-projective.*

2.2. Pseudo-projective modules over semihereditary and hereditary rings

Right semihereditary rings are characterized in the following theorem via the pseudo-projectivity of finitely generated submodules of projective modules, as well as principal right ideals in the endomorphism rings of free modules.

Theorem 2.2.1. *The following conditions are equivalent for a ring R:*

(1) *R is a right semihereditary ring.*

(2) *Every finitely generated submodule of a projective right R-module is pseudo-projective.*

(3) *For any finitely generated free right R-module F, every principal right ideal of $S = \mathrm{End}(F)$ is a pseudo-projective right S-module.*

The following theorem provides a characterization of rings that are both right (left) semihereditary and right (left) S-rings in terms of the pseudo-projectivity of modules.

Theorem 2.2.3. *The following conditions are equivalent for a ring R:*

(1) R is a right semihereditary right S-ring.

(2) R is a left semihereditary left S-ring.

(3) Every finitely generated submodule of a flat right R-module is pseudo-projective.

(4) Every finitely generated submodule of a flat left R-module is pseudo-projective.

We also investigate the characterization of semihereditary and Π-coherent rings via the pseudo-projectivity of finitely generated torsionless right (left) modules and present the following result.

Theorem 2.2.5. *The following statements are equivalent for a ring R:*

(1) R is a semihereditary Π-coherent ring.

(2) Every finitely generated torsionless right R-module is projective.

(3) Every finitely generated torsionless right R-module is pseudo-projective.

(4) Every finitely generated torsionless left R-module is projective.

(5) Every finitely generated torsionless left R-module is pseudo-projective.

Theorem 2.2.1 shows that semihereditary rings can be characterized via the pseudo-projectivity of finitely generated submodules of projective modules. Extending this condition to arbitrary submodules, we obtain a characterization of hereditary rings.

Theorem 2.2.6. *The following conditions are equivalent for a ring R:*

(1) R is right hereditary.

(2) Every submodule of a projective right R-module is pseudo-projective.

(3) Every principal right ideal of $S = \mathrm{End}(F)$ is a pseudo-projective module, for any free right R-module F.

Based on Theorem 2.2.6, by considering flat modules instead of projective ones, we show that hereditary and semiprimitive rings can be characterized via the pseudo-projectivity of all submodules of flat modules.

Theorem 2.2.7. *The following statements are equivalent for a ring R:*

(1) R is a semiprimary hereditary ring.

(2) R is a right hereditary right perfect ring.

(3) R is a left hereditary left perfect ring.

(4) Every submodule of a flat right R-module is pseudo-projective.

(5) Every submodule of a flat left R-module is pseudo-projective.

We establish a ring characterization via the pseudo-projectivity of torsionless modules.

Theorem 2.2.8. *The following conditions are equivalent for a ring R:*

(1) *R is a semiprimary hereditary ring.*

(2) *Every torsionless right R-module is projective.*

(3) *Every torsionless left R-module is projective.*

(4) *Every torsionless right R-module is pseudo-projective.*

(5) *Every torsionless left R-module is pseudo-projective.*

2.3. Pseudo-projective modules over semilocal rings and Artinian principal ideal rings

In the following theorem, we provide a characterization of semilocal rings via the pseudo-projectivity of finitely generated semiprimitive modules.

Theorem 2.3.2. *The following statements are equivalent for a ring R:*

(1) *R is a semilocal ring.*

(2) *Each semiprimitive finitely generated right R-module is Artinian.*

(3) *Each semiprimitive finitely generated right R-module is pseudo-projective.*

It follows from (1) that the conditions (2)–(3) are left–right symmetric.

Faith and Walker characterized quasi-Frobenius rings as those in which every projective module is also injective and vice versa. Based on the results of Fuller, Faith, and Byrd (Theorem 1.3.27), it follows that over Artinian principal ideal rings, quasi-projective modules coincide with dual modules. Motivated by this, we investigate the conditions under which pseudo-projective modules coincide with pseudo-injective modules. The result is presented in the following theorem.

Theorem 2.3.3. *The following conditions are equivalent for a ring R:*

(1) R is an Artinian principal ideal ring.

(2) Every pseudo-projective right R-module is quasi-injective.

(3) Every pseudo-projective right R-module is pseudo-injective.

(4) Every pseudo-injective right R-module is quasi-projective.

(5) Every pseudo-injective right R-module is pseudo-projective.

CHAPTER 3. THE CLASS OF A-RINGS AND GENERALIZATIONS

In this chapter, we investigate rings which every right ideal is automorphism-invariant with I-finite condition. We define a right $\mathfrak{n}$-ring as a ring where every right ideal is nilpotent-invariant. Several properties and structural aspects of this class are also examined.

3.1. Right a-rings

3.1.1. Right I-finite a-rings

The following theorem gives a decomposition of right I-finite a-rings.

Theorem 3.1.4. *Let R be a right I-finite a-ring. Then R is the direct sum of a semisimple Artinian ring and a basic semiperfect ring.*

3.1.2. Right semiperfect a-rings with essential socles

In this section, we assume that R is a right I-finite a-ring in which every minimal right ideal is a right annihilator, and $\mathrm{Soc}(R_R) = \mathrm{Soc}(R_R)$ (denoted by S) is essential in R_R. We study an indecomposable a-ring through a Nakayama permutation and establish a sufficient condition for this class of rings to be quasi-Frobenius.

Theorem 3.1.13. *If R is an indecomposable (not simple) ring with nontrivial idempotents, then R is a quasi-Frobenius ring.*

3.2. Rings in which every right ideal is nilpotent-invariant

3.2.1. Definitions and properties

Definition 3.2.1. A ring R is called a *right* $\mathfrak{n}$*-ring* if every right ideal is nilpotent-invariant.

Example 3.2.3. (1) Every commutative integral domain is a $\mathfrak{n}$-ring. Furthermore, if R is not a field, then R is not an a-ring.

(2) Let $Q = \prod_{i=1}^{\infty} F_i$ where $F_i = K$ is a field with $|\mathbb{K}| > 2$ for all $i \in \mathbb{N}$ and R the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then, R is a $\mathfrak{n}$-ring and not an a-ring.

We also establish a characterization of right $\mathfrak{n}$-rings in connection with essential right ideals.

Proposition 3.2.4. *The following conditions are equivalent for a ring R:*

(1) R is a right n-ring.

(2) Every essential right ideal of R is nilpotent-invariant.

(3) R is right nilpotent-invariant and every essential right ideal of R is a left T-module, where T is a subring of R generated by its identity and its nilpotent elements which are extended to a nilpotent endomorphism of $E(R_R)$.

The non-simple homogeneous components of right $\mathfrak{n}$-rings are described below.

Theorem 3.2.8. *If R is a right n-ring, then R has only finitely many non-simple homogeneous components, and each one of them is of finite index. In particular, each non-simple homogeneous component of R is injective.*

3.2.2. Decomposition of right $\mathfrak{n}$-rings and related rings

The decomposition of right $\mathfrak{n}$-rings into two structurally distinct components, presented below, plays a fundamental role in understanding their internal properties.

Theorem 3.2.13. *Every $\mathfrak{n}$-ring is the direct sum of a square-full semisimple artinian ring and a right square-free ring.*

Based on the internal cancellation property of semisimple Artinian rings and square-free rings, we establish the corresponding property for right $\mathfrak{n}$-rings.

Theorem 3.2.15. *Every right n-ring satisfies the internal cancellation property.*

It is known that every semiprime right a-ring with zero right socle is strongly regular. We further extend this result to regular right $\mathfrak{n}$-rings with no simple submodules.

Theorem 3.2.20. *If R is a regular right n-ring with zero right socle, then R is strongly regular.*

By investigating the characterization of prime right $\mathfrak{n}$-rings with nonzero right socle, we obtain the following result:

Proposition 3.2.22. *Let R be a ring with nonzero right socle. Then, the following conditions are equivalent:*

(1) R is a prime right $\mathfrak{n}$-ring.

(2) R is simple Artinian.

The following theorem concerns the nilpotent-invariance of the matrix ring $\mathbb{M}_n(R)$.

Theorem 3.2.23. *Let $n > 1$ be an integer. Then R is a semisimple Artinian ring if and only if $\mathbb{M}_n(R)$ is a right $\mathfrak{n}$-ring.*

Similar to right a-rings, we conclude that every right $\mathfrak{n}$-ring is stably finite.

Theore 3.2.26. *Every right $\mathfrak{n}$-ring is stably finite.*

3.2.3. Nilpotent-invariant modules over formal triangular matrix rings

The following theorem provides a characterization of nilpotent-invariant modules over formal triangular matrix rings.

Theorem 3.2.27. *Let $K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be a left Artinian ring and (X, Y, f) be a right K-module. If (X, Y, f) is a nilpotent-invariant right K-module then*

(1) Y is a nilpotent-invariant right S-module.

(2) $H = \{x \in X \mid f(x \otimes m) = 0$ for all $m \in M\}$ is a nilpotent-invariant right R-module.

CONCLUSION AND RECOMMENDATIONS

1. Conclusion

The main results obtained in this thesis are summarized as follows:

(1) A characterization of right perfect left coherent rings is established via the pseudo-projectivity of direct products of projective (flat) modules (Theorem 2.1.9).

(2) A characterization of semihereditary S-rings is obtained via the pseudo-projectivity of finitely generated submodules of flat right (left) modules (Theorem 2.2.3).

(3) A characterization of semihereditary Π-coherent rings is derived via the pseudo-projectivity of finitely generated torsionless right (left) modules (Theorem 2.2.5).

(4) A characterization of semiprimary hereditary rings is provided via the pseudo-projectivity of submodules of flat modules or torsionless modules (Theorem 2.2.7 and Theorem 2.2.8).

(5) A characterization of semilocal rings is achieved via the pseudo-projectivity of semiprimitive finitely generated modules (Theorem 2.3.2).

(6) It is shown that, in Artinian principal ideal rings, the class of pseudo-projective modules coincides with the class of pseudo-injective modules (Theorem 2.3.3).

(7) The structure of right I-finite a-rings are described (Theorem 3.1.4).

(8) A characterization of indecomposable right a-rings is given via the Nakayama condition and it is demonstrated that their injective and projective modules coincide (Theorem 3.1.13).

(9) A decomposition of right $\mathfrak{n}$-rings into the direct sum of a square-full semisimple Artinian ring and a right square-free ring is established (Theorem 3.2.13).

(10) The internal cancellation property of right $\mathfrak{n}$-rings is confirmed (Theorem 3.2.15).

(11) A characterization of nilpotent-invariant modules over formal triangular matrix rings is presented (Theorem 3.2.27).

2. Recommendations

Future research will investigate whether every pseudo-projective semiprimitive module implies that the ring is semilocal. Moreover, the structure of $\mathfrak{n}$-rings will be further examined in the context of formal triangular matrix rings.

LIST OF SCIENTIFIC PUBLICATIONS BY THE AUTHOR

(1) Trương Thị Thúy Vân (2023), "Các đặc trưng của PC2-môđun", *Tạp chí Khoa học và Công nghệ - Đại học Đà Nẵng*, 21(3), pp. 58-62.

(2) Nguyen Thi Thu Ha and Truong Thi Thuy Van (2024), "Structure of rings via pseudo-projective modules", *Filomat*, 38(18), pp. 6423-6432.

(3) Truong Cong Quynh and Truong Thi Thuy Van (2024), "On Nilpotent-invariant One-sided Ideals", *Acta Math. Vietnam.*, 49, pp. 115-128.

(4) Truong Thi Thuy Van, Alghamdi, A. M. and Alkinani, A. A. (2024), "On semiperfect a-rings", *Ukr. Math. J.*, 76(6), pp. 1025-1034.